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Partial Derivatives of Repeated Eigenvalues and Their Eigenvectors

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The analysis of inverse problems in linear modeling often require the sensitivities of the eigenvalues and eigenvectors. The calculation of these sensitivities is mathematically related to the corresponding partial derivatives, which do not exist for any parameterization. Inasmuch as eigenvalues and eigenvectors are coupled by the constitutional equation of the general eigenvalue problem, their derivatives are coupled, too. Conditions on the parameterization are derived and formulated as theorems, which ensure the existence of the partial derivatives of the eigenvalues and eigenvectors with respect to these parameters. The application of the theorems is demonstrated by examples.

I. Introduction

MANY engineering optimization problems, for instance, optimal design or model updating, lead to a sensitivity analysis of the eigenvalue problem. In the case of distinct eigenvalues, the partial derivatives of the eigenvalues and eigenvectors with respect to a prechosen parametrization can be calculated. The situation is more complicated in the case of multiple eigenvalues because the associated subbasis of eigenvectors is only defined up to an arbitrary orthonormal matrix. This case has been the subject of many studies in the recent years (see, for instance, Refs. 1–13). As demonstrated by Haug and Rousselet,[8] the (Fréchet) derivatives of the multiple eigenvalues do not exist, in general, for any parametrization in the case of more than one parameter. They suggest the use of the directional (Gateaux) derivatives, which exist for some directions. The question of which parametrization is permissable to ensure the existence of the partial derivatives of the eigenvalues and eigenvectors has not been discussed. A remark on that issue is the purpose of this paper. In Sec. II the problem of calculating the partial derivatives of repeated eigenvalues and the partial derivatives of the corresponding eigenvectors is recalled. Conditions on the existence of the partial derivatives of the eigenvalues and eigenvectors are investigated in Sec. III, accompanied by simple two-dimensional examples. To demonstrate the application of the theorems derived, in the fourth section an example of a three-dimensional elastomechanical model is presented.

II. Recalling the Problem

Consider the general eigenvalue problem

$$AX\Lambda = BX \in \mathbb{R}^{N\times N} \tag{1}$$

with the normalization of the eigenvectors

$$X^\top A X = I_N \tag{2}$$

Both equations are equivalent to

$$A = (XX^\top)^{-1} \tag{3}$$

$$B = (X\Lambda^{-1}X^\top)^{-1} \tag{4}$$

with the real-valued symmetric and positive definite $N \times N$ matrices A and B and the diagonal matrix Λ of the eigenvalues λ_i, $i = 1, \ldots, N$. If $A = A(\boldsymbol{q})$ and $B = B(\boldsymbol{q})$ are given functions of the parameter vector $\boldsymbol{q} \in \mathbb{S} \subseteq \mathbb{R}^m$ such that for each $\boldsymbol{q} \in \mathbb{S}$ decompositions (3) and (4) of $A(\boldsymbol{q})$ and $B(\boldsymbol{q})$ exist, then, of course, the eigenvectors and eigenvalues will depend on $\boldsymbol{q}$. If at parameter vector $\boldsymbol{q}^o$ the first $n \le N$ eigenvalues are equal, i.e.,

$$\Lambda(\boldsymbol{q}^o) = \begin{bmatrix} \ell I_n & 0 \\ 0 & \Gamma \end{bmatrix} \tag{5}$$

then the primary partition $X_1 \in \mathbb{R}^{N\times n}$ of the matrix of eigenvectors $X(\boldsymbol{q}^o) = [X_1, X_2]$ is defined only up to postmultiplication by an $n \times n$ orthogonal matrix Θ. Indeed, from Eqs. (3) and (4), substituting $X_1 \longrightarrow X_1\Theta$ gives

$$X(\boldsymbol{q}^o)X^\top(\boldsymbol{q}^o) = X(\boldsymbol{q}^o)\begin{bmatrix} \Theta & 0 \\ 0 & I_{N-n} \end{bmatrix}\begin{bmatrix} \Theta^\top & 0 \\ 0 & I_{N-n} \end{bmatrix}X^\top(\boldsymbol{q}^o) \tag{6}$$

$$X(\boldsymbol{q}^o)\Lambda^{-1}(\boldsymbol{q}^o)X^\top(\boldsymbol{q}^o) = X(\boldsymbol{q}^o)\begin{bmatrix} \Theta & 0 \\ 0 & I_{N-n} \end{bmatrix}\begin{bmatrix} \ell I_n & 0 \\ 0 & \Gamma \end{bmatrix}^{-1}\begin{bmatrix} \Theta^\top & 0 \\ 0 & I_{N-n} \end{bmatrix}X^\top(\boldsymbol{q}^o) \tag{7}$$

For some parameterizations this redundancy can be used to ensure the existence of the partial derivatives of the eigenvalues at $\boldsymbol{q}^o$. Partial differentiation of Eqs. (3) and (4) with respect to the rth component q_r of $\boldsymbol{q}$ leads to

$$\mathcal{A} := X^\top A_{,r} X = -(Z_r + Z_r^\top) \tag{8}$$

$$\mathcal{B} := X^\top B_{,r} X = -(\Lambda Z_r - \Lambda_{,r} + Z_r^\top \Lambda) \tag{9}$$

where the subscript r denotes the partial derivative with respect to q_r and the matrix Z_r is defined by

$$X_{,r}(\boldsymbol{q}) = X(\boldsymbol{q})Z_r(\boldsymbol{q}), \qquad \forall = 1, \ldots, m \tag{10}$$

Note that Eqs. (8) and (9) imply that, for all $r = 1, \ldots, m$,

$$\mathcal{A} \ \text{symmetric} \Leftrightarrow A_{,r} \ \text{symmetric} \tag{11}$$

$$\mathcal{B} \ \text{symmetric} \Leftrightarrow B_{,r} \ \text{symmetric} \tag{12}$$

Using Eq. (8) to eliminate $Z_r^\top$ in Eq. (9) leads to

$$[Z_r; \Lambda] + \Lambda_{,r} = \mathcal{B} - \mathcal{A}\Lambda =: \mathcal{C} \tag{13}$$

where the commutator product is defined by

$$[A; B] := AB - BA \tag{14}$$

Considering only the diagonal of Eq. (8) leads to

$$(Z_r)_{\text{diag}} = -\tfrac{1}{2}(\mathcal{A})_{\text{diag}} \tag{15}$$

Inserting this result into the diagonal of Eq. (9), $\Lambda_{,r}$ can be calculated, yielding

$$\Lambda_{,r} = (\mathcal{B})_{\text{diag}} - (\mathcal{A}\Lambda)_{\text{diag}} = (\mathcal{C})_{\text{diag}} \tag{16}$$

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This can be done for all $r = 1, \dots, m$ and also at $\boldsymbol{q} = \boldsymbol{q}^o$. The problem is that, at $\boldsymbol{q} = \boldsymbol{q}^o$, Eq. (13) requires the first $n \times n$ diagonal block $\mathcal{C}_r^{11}$ of

$$\mathcal{C}_r(\boldsymbol{q}^o) = \begin{bmatrix} \mathcal{C}_r^{11} & \mathcal{C}_r^{12} \\ \mathcal{C}_r^{21} & \mathcal{C}_r^{22} \end{bmatrix} \tag{17}$$

to be diagonal for all $r = 1, \dots, m$, i.e.,

$$\mathcal{C}_r^{11} := X_1^\top [B_{,r}(\boldsymbol{q}^o) - \ell A_{,r}(\boldsymbol{q}^o)] X_1 = \text{diagonal} \qquad \forall r = 1, \dots, m \tag{18}$$

Only if Eq. (18) holds can the associated part of $\Lambda_{,r}$ be understood as the corresponding partial derivative. Note that, at $\boldsymbol{q} = \boldsymbol{q}^o$, except the first $n \times n$ diagonal block Z_r^{11} of Z_r, the first partial derivative of the eigenvectors can be calculated from Eq. (13). Partitioning of $Z_r(\boldsymbol{q}^o)$ according to the partition of $\Lambda(\boldsymbol{q}^o)$ in Eq. (5) and of $\mathcal{C}_r(\boldsymbol{q}^o)$ in Eq. (17),

$$Z_r(\boldsymbol{q}^o) = \begin{bmatrix} Z_r^{11} & Z_r^{12} \\ Z_r^{21} & Z_r^{22} \end{bmatrix} \tag{19}$$

and using Eq. (13) leads to

$$Z_r^{12} = \mathcal{C}_r^{12}(\Gamma - \ell I_{N-n})^{-1} \tag{20}$$

$$Z_r^{21} = -(\Gamma - \ell I_{N-n})^{-1}\mathcal{C}_r^{21} \tag{21}$$

and for $N - n > 1$

$$(Z_r^{22})_{ij} = \frac{(\mathcal{C}_r^{22})_{ij}}{\gamma_j - \gamma_i}, \qquad i \neq j, \qquad i, j \in \{1, \dots, N-n\} \tag{22}$$

where $\gamma_i = (\Gamma)_{ii}$. The expression in Eq. (22) is equivalent to the coefficients of the standard series approach for nonrepeated eigenvalues. Note that the diagonal of Z_r^{11} and of Z_r^{22} is given by Eq. (15). To calculate the off-diagonal part $(Z_r^{11})_{\text{off}}$ of Z_r^{11} the second partial derivatives of the eigenvalue problem are needed. Here the subscript off denotes a matrix with a zero diagonal.

Differentiating Eqs. (8) and (9) with respect to q_s yields

$$\begin{aligned} \mathcal{A}_{rs} &:= X^\top A_{,r,s} X \\ &= -(Z_{,r,s} + Z_{,r,s}^\top) - Z_s^\top \mathcal{A}_r - \mathcal{A}_r Z_s \end{aligned} \tag{23}$$

$$\begin{aligned} \mathcal{B}_{rs} &:= X^\top B_{,r,s} X \\ &= -(\Lambda_{,s} Z_r + \Lambda Z_{,r,s} - \Lambda_{,r,s} + Z_{,r,s}^\top \Lambda + Z_r^\top \Lambda_{,s}) \\ &\quad - Z_s^\top \mathcal{B}_r - \mathcal{B}_r Z_s \end{aligned} \tag{24}$$

where according to Eq. (10)

$$X_{,r,s} = X(Z_s Z_r + Z_{,r,s}) \tag{25}$$

Again using Eq. (8) to eliminate $Z_r^\top$ and $Z_s^\top$ leads to

$$\mathcal{A}_{rs} = -(Z_{,r,s} + Z_{,r,s}^\top) + [Z_s; \mathcal{A}_r] + \mathcal{A}_s \mathcal{A}_r \tag{26}$$

$$\begin{aligned} \mathcal{B}_{rs} &= -(\Lambda Z_{,r,s} + Z_{,r,s}^\top \Lambda) + \Lambda_{,r,s} + \mathcal{A}_r \Lambda_{,s} + \mathcal{A}_s \mathcal{B}_r \\ &\quad + [Z_r; \Lambda_{,s}] + [Z_s; \mathcal{B}_r] \end{aligned} \tag{27}$$

Inserting $Z_{,r,s}^\top$ from Eq. (26) into Eq. (27) and using Eq. (13) finally yields

$$\begin{aligned} Q_{rs} &:= \mathcal{B}_{rs} - \mathcal{A}_{rs}\Lambda - \mathcal{A}_s \mathcal{C}_r - \mathcal{A}_r \mathcal{C}_s \\ &= [Z_{,r,s}; \Lambda] + [Z_r; \Lambda_{,s}] + [Z_s; \mathcal{C}_r] + \Lambda_{,r,s} \end{aligned} \tag{28}$$

On the right-hand side of this equation the diagonal of the first commutator vanishes at $\boldsymbol{q} = \boldsymbol{q}^o$ and the diagonal of the second commutator is zero because $\Lambda_{,s}$ is diagonal. Only the off-diagonal elements of Z_r^{11}, $r = 1, \dots, m$, are unknown. But this block does not occur in the diagonal of the last commutator on the right-hand side of Eq. (28). That can be verified by recalling that with reference to Eq. (18) the first diagonal block of $\mathcal{C}_r$ is diagonal, i.e.,

$$\mathcal{C}_r^{11} = \Lambda_{,r}^1 \tag{29}$$

with the first partition $\Lambda_{,r}^1$ of the eigenvalue derivatives given by

$$\Lambda_{,r} = \begin{bmatrix} \Lambda_{,r}^1 & 0 \\ 0 & \Lambda_{,r}^2 \end{bmatrix} \tag{30}$$

Only the off-diagonal blocks of the last commutator depend on Z_s^{11}. Thus, the second partial derivatives of the eigenvalues can be calculated from the diagonal of Eq. (28), yielding

$$\Lambda_{,r,s} = (Q_{rs})_{\text{diag}} - \begin{bmatrix} (Z_s^{12}\mathcal{C}_r^{21} - \mathcal{C}_r^{12} Z_s^{21})_{\text{diag}} & 0 \\ 0 & (Z_s^{21}\mathcal{C}_r^{12} - \mathcal{C}_r^{21} Z_s^{12} + [Z_s^{22}; \mathcal{C}_r^{22}])_{\text{diag}} \end{bmatrix} \tag{31}$$

It remains to derive conditions that enable the calculation of the unknown off-diagonal of Z_r^{11}. Because the first partition of Eq. (28) does not depend on $Z_{r,s}$ it can be used to calculate $(Z_r^{11})_{\text{off}}$, i.e.,

$$(Q_{rs}^{11})_{\text{off}} = [(Z_r^{11})_{\text{off}}; \Lambda_{,s}^1] + [(Z_s^{11})_{\text{off}}; \Lambda_{,r}^1] + (Z_s^{12}\mathcal{C}_r^{21} - \mathcal{C}_r^{12} Z_s^{21})_{\text{off}} \tag{32}$$

For distinct eigenvalue derivatives, Eq. (32) allows the calculation of $(Z_r^{11})_{\text{off}}$ using, for instance, the diagonal $r = s$, which yields with reference to Eq. (21) for $i \neq k$:

$$(Z_r^{11})_{ik} = \frac{1}{2(\lambda_{k,r} - \lambda_{i,r})}\left[Q_{rr}^{11} - 2\mathcal{C}_r^{12}(\Gamma - \ell I_{N-n})^{-1}\mathcal{C}_r^{21}\right]_{ik} \tag{33}$$

Of course, the consistency of Eq. (32) with this result has to be checked for the remaining equations resulting from $r \neq s$ because the coupling of the sensitivities of the eigenvalues and eigenvectors means that their existence is coupled, too. In the following section, conditions for the existence of the first partial derivatives will be inferred and their continuity is investigated.

III. Existence of Eigenvalue and Eigenvector Sensitivities

A. Two Theorems on the Existence of the Partial Derivatives of Repeated Eigenvalues

As already pointed out, the eigenvectors related to the multiple eigenvalues, in general, are not unique. In some cases it is possible to choose the orthonormal matrix Θ in such a way that it diagonalizes the matrices $\mathcal{C}_r^{11}$ for all $r = 1, \dots, m$. Thus, the existence of the partial derivatives of the eigenvalues depends on the choice of the parameterization.

Theorem 1: A necessary and sufficient condition of the existence of the partial derivatives of Λ at $\boldsymbol{q}^o$ is that

$$[\mathcal{C}_r^{11}; \mathcal{C}_s^{11}] = 0 \qquad \forall r, s = 1, \dots, m \tag{34}$$

From basic algebra (for further reading see, for instance, Refs. 14–16) it is clear that for two matrices having the same orthonormal eigenvectors the commutator vanishes. On the other hand, if the commutator of two matrices vanishes they have the same diagonalizing matrix. If both matrices are symmetric [see Eqs. (11) and (12)], the diagonalizing matrix is orthonormal. The fact that there exist a maximum n linearly independent commutating $n \times n$ matrices leads to the following theorem.

Theorem 2: Denoting $\text{cs}(\mathcal{C}_r^{11})$ the n^2-dimensional vector containing the sequence of the column vectors of $\mathcal{C}_r^{11}$, then a necessary condition for the exsistence of the first partial derivative of the eigenvalues at $\boldsymbol{q} = \boldsymbol{q}^o$ is

$$\text{rank}([\text{cs}(\mathcal{C}_1^{11}), \dots, \text{cs}(\mathcal{C}_m^{11})]) \leq n \tag{35}$$

The conditions of Theorems 1 and 2 enable a given parameterization to be tested. If either theorem is violated, further computations are in vain because the derivatives will not exist at $\boldsymbol{q} = \boldsymbol{q}^o$. Moreover, if the parameterization is permissible in the specified sense, at $\boldsymbol{q} = \boldsymbol{q}^o$ the eigenvalues and eigenvectors and their first partial derivatives are continuous if $A(\boldsymbol{q})$ and $B(\boldsymbol{q})$ and their first derivatives are continuous. As a consequence of the continuity of the first partial derivatives, the order of differentiation within the second partial derivatives is arbitrary. Interchanging $r \leftrightarrow s$, for example, in Eq. (28) and using $\mathcal{B}_{rs} = \mathcal{B}_{sr}$ as well as $\mathcal{A}_{rs} = \mathcal{A}_{sr}$ lead to

$$
\begin{aligned}
0 &= Q_{rs} - Q_{sr} \\
&= [Z_{r,s} - Z_{s,r}; \Lambda] + [Z_r; \Lambda_{,s}] - [Z_s; \Lambda_{,r}] + [Z_s; \mathcal{C}_r] \\
&\quad - [Z_r; \mathcal{C}_s] + \Lambda_{,r,s} - \Lambda_{,s,r} \\
&= [Z_{r,s} - Z_{s,r}; \Lambda] + [Z_r; \Lambda_{,s}] - [Z_s; \Lambda_{,r}] + [Z_s; [Z_r; \Lambda]] \\
&\quad + [Z_s; \Lambda_{,r}] - [Z_r; [Z_s; \Lambda]] - [Z_r; \Lambda_{,s}] + \Lambda_{,r,s} - \Lambda_{,s,r} \\
&= [Z_{r,s} - Z_{s,r}; \Lambda] + \Lambda_{,r,s} - \Lambda_{,s,r} - [Z_r; [Z_s; \Lambda]] \\
&\quad + [Z_s; [Z_r; \Lambda]]
\end{aligned}
\tag{36}
$$

where for $\mathcal{C}_r$ and $\mathcal{C}_s$ the corresponding left-hand sides of Eq. (13) have been inserted. Interchanging the arguments in the last commutator of Eq. (36) and using the Bianci identity

$$[A; [B; C]] + [B; [C; A]] + [C; [A; B]] = 0 \tag{37}$$

to rewrite the last two commutators in Eq. (36) yields

$$[Z_{r,s} - Z_{s,r} - [Z_r; Z_s]; \Lambda] + \Lambda_{,r,s} - \Lambda_{,s,r} = 0 \tag{38}$$

In general, this equation holds if

$$\Lambda_{,r,s} = \Lambda_{,s,r} \tag{39}$$

$$Z_{r,s} - Z_{s,r} = [Z_r; Z_s] \tag{40}$$

The first equation expresses the arbitrariness of the order of differentiation of the eigenvalues and is equivalent with the continuity of their first partial derivative. The second equation is, with reference to Eq. (25), equivalent with the continuity of the first partial derivatives of the eigenvectors. Before corresponding conditions for the existence of the partial derivative of the eigenvectors will be derived the following academical example of Seyranian et al.[13] (see also Ref. 7, p. 158) shows that not for any parameterization do the partial derivatives of the eigenvalues exist.

For the example with a linear parameterization, let $A = I_2$, and for $\boldsymbol{q} = (q_1, q_2)^\top$, let

$$B(\boldsymbol{q}) := I_2 + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} q_1 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} q_2 \tag{41}$$

which leads to the repeated eigenvalue $1 = \lambda_1 = \lambda_2$ at $\boldsymbol{q} = 0$. Theorem 2 does hold in this case because

$$\operatorname{rank}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = 2 = n \tag{42}$$

Checking Theorem 1 by using $\mathcal{C}_r = \mathcal{B}_r = B_{,r}$ leads to

$$[B_1; B_2] = \left[\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \neq 0 \tag{43}$$

Thus, the given parameterization (41) is not permissible. Indeed, the calculation of the eigenvalues leads to

$$\lambda(\boldsymbol{q}) = 1 + \tfrac{3}{2} q_1 \pm w(\boldsymbol{q}) \tag{44}$$

where

$$w(\boldsymbol{q}) := \sqrt{(q_1/2)^2 + q_2^2} \tag{45}$$

The first partial derivatives are

$$\frac{\partial \lambda}{\partial \boldsymbol{q}} = \frac{3}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \frac{1}{4w}\begin{pmatrix} q_1 \\ 4q_2 \end{pmatrix} \tag{46}$$

and the second partial derivatives turn out to be

$$\frac{\partial^2 \lambda}{\partial \boldsymbol{q} \partial \boldsymbol{q}^\top} = \pm \frac{1}{4w}\begin{bmatrix} -q_2^2 & q_1 q_2 \\ q_1 q_2 & -q_1^2 \end{bmatrix} \tag{47}$$

Obviously the limits of the derivatives as $\boldsymbol{q} \to 0$ do not exist.

B. Two Theorems on the Existence of the Partial Derivatives of Eigenvectors

Although it is popular (see, for instance, Ref. 5 or 9) to use the diagonal $r = s$ of Eq. (32) only to calculate $(Z_r^{11})_{\text{off}}$, it may be that this solution violates the remaining equations resulting from $r \neq s$. Thus, it remains to investigate Eq. (32) and to assure its consistency. It is sufficient to show that the system of equations

$$
\begin{aligned}
(G_{rs})_{\text{off}} &= \left[(Z_r^{11})_{\text{off}}; \Lambda^1_{,s}\right] + \left[(Z_s^{11})_{\text{off}}; \Lambda^1_{,r}\right] \\
&= (Z_r^{11})_{\text{off}} \Lambda^1_{,s} - \Lambda^1_{,s} (Z_r^{11})_{\text{off}} + (Z_s^{11})_{\text{off}} \Lambda^1_{,r} - \Lambda^1_{,r} (Z_s^{11})_{\text{off}} \\
&\qquad \forall r, s = 1, \ldots, m
\end{aligned}
\tag{48}
$$

has unique solutions $(Z_r^{11})_{\text{off}}$, $r = 1, \ldots, m$, for given diagonal matrices $\Lambda^1_{,r}$, having distinct elements, i.e., $\lambda_{i,r} \neq \lambda_{k,r}$, $i \neq k$, for $i, k \in \{1, \ldots, n\}$ for each $r = 1, \ldots, m$. The matrix G_{rs} in Eq. (48) represents all known terms of Eq. (32), i.e.,

$$(G_{rs})_{\text{off}} := (Q_{rs}^{11})_{\text{off}} - (Z_s^{12} \mathcal{C}_r^{21} + \mathcal{C}_r^{12} Z_s^{21})_{\text{off}} \tag{49}$$

where Z_s^{12} and Z_s^{21} are known from Eqs. (20) and (21). Because the diagonal part of Eq. (48) is identically zero it represents a coupled system of $n(n-1)$ equations. Writing one equation for the element g_{ikrs} of G_{rs} in row i and column k, where $i \neq k$ for $i, k \in \{1, \ldots, n\}$ and denoting the corresponding element of Z_r^{11} by z_{ikr} Eq. (48) reads

$$g_{ikrs} = (\lambda_{k,s} - \lambda_{i,s}) z_{ikr} + (\lambda_{k,r} - \lambda_{i,r}) z_{iks} \tag{50}$$

Expanding this equation to m^2 equations resulting from $r, s = 1, \ldots, m$, leads to

$$
\underbrace{\begin{bmatrix} g_{ik11} & \ldots & g_{ik1m} \\ \vdots & \ddots & \vdots \\ g_{ikm1} & \ldots & g_{ikmm} \end{bmatrix}}_{=:\ \mathcal{G}_{ik}}
= \underbrace{\begin{pmatrix} z_{ik1} \\ \vdots \\ z_{ikm} \end{pmatrix}}_{=:\ \boldsymbol{z}_{ik}} \underbrace{(\lambda_{k,1} - \lambda_{i,1}, \ldots, \lambda_{k,m} - \lambda_{i,m})}_{=:\ \boldsymbol{h}_{ik}^\top} + \boldsymbol{h}_{ik} \boldsymbol{z}_{ik}^\top \in \mathbb{R}^{m \times m}
\tag{51}
$$

Because the right-hand side of this equation is symmetric, the left-hand side has to be symmetric, too. Moreover, the ranks of both sides have to be the same, which is equivalent to the condition

$$\operatorname{rank}(\mathcal{G}_{ik}) = \operatorname{rank}(\boldsymbol{z}_{ik} \boldsymbol{h}_{ik}^\top + \boldsymbol{h}_{ik} \boldsymbol{z}_{ik}^\top) \tag{52}$$

Of course, Eq. (51) has a unique solution only if

$$N_{ik} \mathcal{G}_{ik} N_{ik} = 0 \tag{53}$$

where the symmetric matrix N_{ik} is the orthogonal projector (see, for instance, Ref. 17) into the $(m-1)$-dimensional orthogonal complement of the one-dimensional subspace spanned by $\boldsymbol{h}_{ik}$, i.e.,

$$N_{ik} := I_m - P_{ik} \tag{54}$$

where

$$P_{ik} := \frac{\boldsymbol{h}_{ik} \boldsymbol{h}_{ik}^\top}{\|\boldsymbol{h}_{ik}\|^2} \tag{55}$$

is the orthogonal projector into the subspace spanned by $\boldsymbol{h}_{ik} \in \mathbb{R}^m$. To see this, one can multiply Eq. (51) by $\boldsymbol{h}_{ik}$, yielding

$$\begin{aligned}\mathcal{G}^{ik}\boldsymbol{h}_{ik} &= \boldsymbol{z}_{ik}\|\boldsymbol{h}_{ik}\|^2 + \boldsymbol{h}_{ik}\boldsymbol{z}_{ik}^\top\boldsymbol{h}_{ik} \\ &= \left(\|\boldsymbol{h}_{ik}\|^2 I_m + \boldsymbol{h}_{ik}\boldsymbol{h}_{ik}^\top\right)\boldsymbol{z}_{ik}\end{aligned} \tag{56}$$

For $\|\boldsymbol{h}_{ik}\| \neq 0$, the matrix on the right-hand side of Eq. (56) is nonsingular. The unique solution of Eq. (56) is given by

$$\begin{aligned}\boldsymbol{z}_{ik} &= \frac{1}{\|\boldsymbol{h}_{ik}\|^4}\left(\|\boldsymbol{h}_{ik}\|^2 I_m - \frac{1}{2}\boldsymbol{h}_{ik}\boldsymbol{h}_{ik}^\top\right)\mathcal{G}^{ik}\boldsymbol{h}_{ik} \\ &= \frac{1}{\|\boldsymbol{h}_{ik}\|^2}\left(I_m - \frac{1}{2}P_{ik}\right)\mathcal{G}^{ik}\boldsymbol{h}_{ik}\end{aligned} \tag{57}$$

Of course, in general this solution is not a solution of Eq. (51). Inserting $\boldsymbol{z}_{ik}$ from Eq. (57) into Eq. (51) yields

$$\begin{aligned}\mathcal{G}^{ik} &= \left(I_m - \tfrac{1}{2}P_{ik}\right)\mathcal{G}^{ik}P_{ik} + P_{ik}\mathcal{G}^{ik}\left(I_m - \tfrac{1}{2}P_{ik}\right) \\ &= \mathcal{G}^{ik}P_{ik} + P_{ik}\mathcal{G}^{ik} - P_{ik}\mathcal{G}^{ik}P_{ik}\end{aligned} \tag{58}$$

The latter equation is equivalent to

$$0 = \mathcal{G}^{ik} - \mathcal{G}^{ik}P_{ik} - P_{ik}\mathcal{G}^{ik} + P_{ik}\mathcal{G}^{ik}P_{ik} = N_{ik}\mathcal{G}^{ik}N_{ik} \tag{59}$$

Summarizing what has been said in this section leads to the following.

Theorem 3: Necessary conditions for the existence of a unique solution of Eq. (51) are

$$\mathcal{G}^{ik} = \mathcal{G}^{ik\top} \tag{60}$$

$$\operatorname{rank}(\mathcal{G}^{ik}) \leq 2 \tag{61}$$

Moreover, Eq. (32) is consistent with the result given in Eq. (33) if and only if

$$N_{ik}\mathcal{G}^{ik}N_{ik} = 0 \qquad \forall i \neq k, \qquad i, k \in \{1, \ldots, n\} \tag{62}$$

where $\mathcal{G}^{ik}$ and N_{ik} are defined in Eqs. (51–55).

Of course, this theorem leads to restrictions on the given parameterization. Inserting the expressions given in Eqs. (20) and (21) into Eq. (49), and using Eq. (28) with partitions for $\mathcal{A}_r$, $\mathcal{B}_{rs}$, $\mathcal{A}_{rs}$ and for $\mathcal{B}_r$ corresponding to that of Z_r and $\mathcal{C}_r$ defined in Eqs. (19) and (17), respectively, yields

$$G_{rs} = \mathcal{B}^{11}_{rs} - \ell\mathcal{A}^{11}_{rs} - \mathcal{A}^{11}_{s}\Lambda^1_{,r} - \mathcal{A}^{11}_{r}\Lambda^1_{,s} - \mathcal{C}^{21\top}_{s}D\mathcal{C}^{21}_{r} - \mathcal{C}^{21\top}_{r}D\mathcal{C}^{21}_{s} \tag{63}$$

where $D := (\Gamma - \ell I_{N-n})^{-1}$. Considering the element in row i and column k of Eq. (63) leads to

$$g_{ikrs} = b_{ikrs} - \ell a_{ikrs} - a_{iks}\lambda_{k,r} - a_{ikr}\lambda_{k,s} - \boldsymbol{c}_{is}^\top D\boldsymbol{c}_{kr} - \boldsymbol{c}_{ir}^\top D\boldsymbol{c}_{ks} \tag{64}$$

where b_{ikrs}, a_{ikrs}, and a_{ikr} denoting the elements in the ith row and in the kth column of the matrices $\mathcal{B}^{11}_{rs}$, $\mathcal{A}^{11}_{rs}$, and $\mathcal{A}^{11}_{r}$, respectively, and $\boldsymbol{c}_{ir}$ is the ith column vector of $\mathcal{C}^{21}_r$. Writing Eq. (64) for all rows $r = 1, \ldots, m$ and for all columns $s = 1, \ldots, m$ yields

$$\begin{aligned}\mathcal{G}^{ik} = &\underbrace{\begin{bmatrix} b_{ik11} & \ldots & b_{ik1m} \\ \vdots & \ddots & \vdots \\ b_{ikm1} & \ldots & b_{ikmm}\end{bmatrix}}_{=:\,B_{ik}} - \ell\underbrace{\begin{bmatrix} a_{ik11} & \ldots & a_{ik1m} \\ \vdots & \ddots & \vdots \\ a_{ikm1} & \ldots & a_{ikmm}\end{bmatrix}}_{=:\,A_{ik}} \\ &- \underbrace{\begin{pmatrix}\lambda_{k,1} \\ \vdots \\ \lambda_{k,m}\end{pmatrix}}_{=:\,\boldsymbol{f}_k}\underbrace{(a_{ik1}, \ldots, a_{ikm})}_{=:\,\boldsymbol{a}_{ik}^\top} - \boldsymbol{a}_{ik}\boldsymbol{f}_k^\top - \underbrace{\begin{bmatrix}\boldsymbol{c}_{k1}^\top \\ \vdots \\ \boldsymbol{c}_{km}^\top\end{bmatrix}}_{C_k^\top} DC_i - C_i^\top DC_k\end{aligned} \tag{65}$$

Obviously, the matrices $\mathcal{G}^{ik}$ are symmetric for all $i \neq k \in \{1, \ldots, n\}$. Thus, the first of the necessary conditions of Theorem 3 does hold. Although Eq. (65) is related to Eq. (53) via $\boldsymbol{h}_{ik} = \boldsymbol{f}_k - \boldsymbol{f}_i$, neither the rank condition nor the sufficient condition formulated in Theorem 3 does necessarily hold for any parameterization. Even in the case of a linear parameterization of the matrix B, Eq. (65) becomes

$$\mathcal{G}^{ik} = -C_k^\top DC_i - C_i^\top DC_k \tag{66}$$

which may be of rank $m > 2$ for some $i \neq k$, $i, k \in \{1, \ldots, n\}$. The sufficient condition formulated in Theorem 3 should be checked numerically rather than analytically. The analytical expression of how the projector N_{ik} acts on $\mathcal{G}^{ik}$ is rather difficult. A better insight can be achieved by translating the effect of N_{ik} on G_{rs} rather than on $\mathcal{G}^{ik}$. Of course, in the light of Eq. (53) the effect of N_{ik} on $\mathcal{G}^{ik}$ is related to the effect of the projector P_{ik}:

$$0 = N_{ik}\mathcal{G}^{ik}N_{ik} = \mathcal{G}^{ik} - \mathcal{G}^{ik}P_{ik} - P_{ik}\mathcal{G}^{ik} + P_{ik}\mathcal{G}^{ik}P_{ik} \tag{67}$$

Considering only the element in row r and in column s of Eq. (67) and expanding this expression for all $i \neq k \in \{1, \ldots, n\}$ leads to

$$0 = (G_{rs})_{\text{off}} - H \odot \left([S_s; \Lambda^1_{,r}] + [S_r; \Lambda^1_{,s}]\right) + H \odot [[T; \Lambda^1_{,r}]; \Lambda^1_{,s}] \tag{68}$$

where $\odot$ denotes the Hadamard product, which is the componentwise product of two matrices having the same size,[17] and

$$S_r := \sum_{s=1}^{m}[G_{sr}; \Lambda^1_{,s}] \tag{69}$$

$$T := \sum_{s=1}^{m}[\Lambda^1_{,s}; S_s] \tag{70}$$

$$(H)_{ik} := \left[\sum_{r=1}^{m}(\lambda_{k,r} - \lambda_{i,r})^2\right]^{-1} \tag{71}$$

This leads to the formulation of the following theorem.

Theorem 4: Equation (32) is consistent with the result given in Eq. (33) if and only if

$$(G_{rs})_{\text{off}} = H \odot \left([S_s; \Lambda^1_{,r}] + [S_r; \Lambda^1_{,s}]\right) + H \odot [[T; \Lambda^1_{,r}]; \Lambda^1_{,s}] \tag{72}$$

$\forall r, s = 1, \ldots, m$, where S_r, T, and H are defined in Eqs. (69–71).

Of course, Theorem 3 is equivalent to Theorem 4 but for a numerical check either one of them may be used. Before a three-dimensional example is presented (in the next section), the following academical example of a nonlinear parameterization shows that, though the partial derivatives of the eigenvalues exist, the partial derivatives of the eigenvectors do not exist in general.

For the example with a nonlinear parameterization, let $A = I_2$ and define $B(\boldsymbol{q})$ with $\boldsymbol{q} \in \mathbb{R}^2$ by

$$B(\boldsymbol{q}) := I_2 + \begin{bmatrix}1 & 0 \\ 0 & 2\end{bmatrix}q_1 + \begin{bmatrix}0 & -q_2 \\ -q_2 & 1\end{bmatrix}q_2 \tag{73}$$

which leads at $\boldsymbol{q} = 0$ to eigenvalues $\lambda_1 = \lambda_2 = 1$. A brief calculation shows

$$\mathcal{C}_1 = \mathcal{B}_1 = B_{,1} = \Lambda_{,1} = \begin{bmatrix}1 & 0 \\ 0 & 2\end{bmatrix} \tag{74}$$

$$\mathcal{C}_2 = \mathcal{B}_2 = B_{,2} = \Lambda_{,2} = \begin{bmatrix}0 & 0 \\ 0 & 1\end{bmatrix} \tag{75}$$

Thus, due to Theorems 1 and 2, the partial derivatives of the eigenvalues exist. To check Theorem 3 or 4, one has to calculate the following quantities:

$$\mathcal{B}_{11} = B_{,1,1} = 0 = Q_{11} \tag{76}$$

$$\mathcal{B}_{12} = \mathcal{B}_{21} = B_{,1,2} = 0 = Q_{12} = Q_{21} \tag{77}$$

$$\mathcal{B}_{22} = B_{,2,2} = Q_{22} = -\begin{bmatrix}0 & 2 \\ 2 & 0\end{bmatrix} \tag{78}$$

A brief calculation shows that $\mathcal{G}_{ik} = 0$ for all but one pair of indices $(i, k) = (1, 2)$, yielding

$$\mathcal{G}_{12} = -2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{79}$$

On the other hand,

$$\boldsymbol{h}_{12} = \begin{pmatrix} \lambda_{2,1} - \lambda_{1,1} \\ \lambda_{2,2} - \lambda_{1,2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{80}$$

and, therefore,

$$N_{12} = I_2 - \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{81}$$

Finally, to check Theorem 3, one has to calculate

$$N_{12}\mathcal{G}_{12}N_{12} = -\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{82}$$

$$= \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \neq 0 \tag{83}$$

Thus, the partial derivatives of the eigenvectors do not exist. A direct calculation by using the diagonal [see Eq. (33)] without checking the consistency would lead to the incorrect results

$$Z_1 = 0 \tag{84}$$

$$Z_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{85}$$

In the next section, a three-dimensional example is investigated concerning permissible linear parameterizations.

IV. Example

To demonstrate the application of the theorems, the spring-mass model depicted in Fig. 1, which corresponds to the example presented by Friswell,[5] is used. The mass matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{86}$$

is assumed to be constant and the stiffness matrix is parameterized by

$$\begin{aligned} B(q) = {} & \underbrace{\boldsymbol{e}_1\boldsymbol{e}_1^\top}_{=:\,B_1} q_1 + 8\underbrace{\boldsymbol{e}_2\boldsymbol{e}_2^\top}_{=:\,B_2} q_2 + \underbrace{\boldsymbol{e}_3\boldsymbol{e}_3^\top}_{=:\,B_3} q_3 \\ & + \underbrace{2(\boldsymbol{e}_1 - \boldsymbol{e}_2)(\boldsymbol{e}_1 - \boldsymbol{e}_2)^\top}_{=:\,B_4} q_4 + \underbrace{2(\boldsymbol{e}_2 - \boldsymbol{e}_3)(\boldsymbol{e}_2 - \boldsymbol{e}_3)^\top}_{=:\,B_5} q_5 \\ & + \underbrace{(\boldsymbol{e}_1 - \boldsymbol{e}_3)(\boldsymbol{e}_1 - \boldsymbol{e}_3)^\top}_{=:\,B_6} q_6 \end{aligned} \tag{87}$$

which corresponds (see Fig. 1) to

$$(k_1, k_2, k_3, k_4, k_5, k_6) = (q_1, 8q_2, q_3, 2q_4, 2q_5, q_6) \tag{88}$$

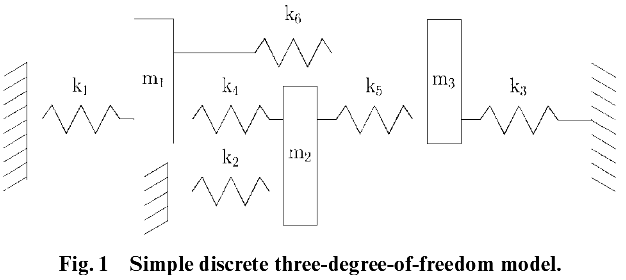

Fig. 1 Simple discrete three-degree-of-freedom model.

Table 1 Generators of the symmetric dyads of $\mathcal{C}_r$

$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$
a	0	$-a$	$2a$	$2a$	$2a$
b	$-8b$	b	$4b$	$-4b$	0
$2c$	$8c$	$2c$	$2c$	$-2c$	0

Table 2 Upper off-diagonal elements of the skew-symmetric commutators: $[\mathcal{C}_r^{11}; \mathcal{C}_s^{11}] \in \mathbb{R}^{2\times 2}$ for tuples (r, s) with $r < s$

(r, s)	$[\mathcal{C}_r^{11}; \mathcal{C}_s^{11}]/ab$	(r, s)	$[\mathcal{C}_r^{11}; \mathcal{C}_s^{11}]/ab$
(1, 2)	32/3	(2, 6)	0
(1, 3)	−1	(3, 4)	2
(1, 4)	10/3	(3, 5)	−10/3
(1, 5)	−2	(3, 6)	2
(1, 6)	−2	(4, 5)	32/3
(2, 3)	32/3	(4, 6)	−16
(2, 4)	−256/3	(5, 6)	16
(2, 5)	256/3		

At $\boldsymbol{q} = (0, 1, 0, 1, 1, 1)^\top$, a repeated eigenvalue $\lambda_1 = \lambda_2 = 4$ occurs. The associated eigenvectors are

$$\boldsymbol{x}_1 = a \cdot (1, 0, -1)^\top, \qquad \boldsymbol{x}_2 = b \cdot (1, -1, 1)^\top \tag{89}$$

The remaining eigenvalue is $\lambda_3 = 1$ with the eigenvector

$$\boldsymbol{x}_3 = c \cdot (2, 1, 2)^\top \tag{90}$$

The normalization constants are $a := 1/\sqrt{2}$, $b := 1/\sqrt{6}$, and $c := 1/\sqrt{12}$. Because A does not depend on the parameters $\mathcal{C}_r = \mathcal{B}_r$ and because all submatrices B_r are symmetric generated by a single vector, each $\mathcal{C}_r$ also is symmetric and generated by one vector only. The generating vectors are listed in Table 1. For example,

$$\mathcal{C}_1 = (a, b, 2c)^\top(a, b, 2c) \tag{91}$$

To check Theorems 1 and 2, the matrices $\mathcal{C}_r^{11} \in \mathbb{R}^{2\times 2}$ have to be calculated for all $r = 1, \ldots, 6$. They are generated by the two-dimensional vectors containing the first two components of the generators listed in Table 1, for instance,

$$\mathcal{C}_1^{11} = (a, b)^\top(a, b) \tag{92}$$

Theorem 2 yields

$$\operatorname{rank}\left(\begin{bmatrix} a^2 & 0 & a^2 & 1 & 1 & 2 \\ ab & 0 & -ab & 4ab & -4ab & 0 \\ ab & 0 & -ab & 4ab & -4ab & 0 \\ b^2 & 8b^2 & b^2 & 8b^2 & 8b^2 & 0 \end{bmatrix}\right) = 4 > 2 = n \tag{93}$$

Thus, the partial derivatives of the eigenvalues will not exist for the complete parameterization as defined by Eq. (87). To answer the question of a permissible parameterization Theorem 1 has to be checked. This requires the calculation of 15 commutators. The resulting matrices are skew symmetric. The corresponding upper off-diagonal elements are listed in Table 2 for all $r < s$. Of the six matrices only the two corresponding to parameters q_2 and q_6 commute. Thus, an orthogonal matrix $\Theta \in \mathbb{R}^{2\times 2}$ will only exist for the parameterization

$$B(q_1, q_2) = 2\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + 8\boldsymbol{e}_2\boldsymbol{e}_2^\top q_1 + (\boldsymbol{e}_1 - \boldsymbol{e}_3)(\boldsymbol{e}_1 - \boldsymbol{e}_3)^\top q_2 \tag{94}$$

where the parameters have been changed according to $(q_2, q_6) \longrightarrow (q_1, q_2)$. Using this new parameterization to calculate the first derivative of the eigenvalues from Eq. (13) at $q_1 = q_2 = 0$ it turned out that $\mathcal{C}_r$ is already diagonal, i.e., $\Theta = I_2$. The eigenvalue derivatives are

$$\frac{\partial \Lambda}{\partial q_1} = \frac{2}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{95}$$

$$\frac{\partial \Lambda}{\partial q_2} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{96}$$

Because the inertia matrix A does not depend on the parameters from Eq. (8), it follows that Z_r is skew symmetric and, thus, $(Z_r)_{ii} = 0$ for $r = 1, 2$ and for $i = 1, \ldots, N$. Equations (20) and (21) lead to

$$(Z_r)_{13} = 0 \qquad \forall = 1, 2 \tag{97}$$

$$(Z_1)_{23} = \tfrac{8}{3}bc \tag{98}$$

$$(Z_2)_{23} = 0 \tag{99}$$

To determine the off-diagonal part of $(Z_r)_{12}$, Eq. (32) can be used. For the example considered here, Q_{rs} and $\mathcal{A}$ are zero and Eq. (32) is consistent with the result

$$(Z_r)_{12} = 0 \qquad \forall = 1, 2 \tag{100}$$

The result $X_{,2} = XZ_2 = 0$ means that a first-order approximation change in the stiffness k_6 do not affect the eigenvectors.

Using Eq. (31) to calculate the second derivatives of the eigenvalues this particular example leads to

$$(\Lambda_{,r,s})_{ii} = ([\mathcal{C}_r, Z_s])_{ii} \tag{101}$$

which yields

$$\frac{\partial^2 \Lambda}{\partial q_1^2} = \frac{16}{27}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{102}$$

All other derivatives turned out to be zero.

To compare these results with those presented by Friswell,[5] the derivatives of the eigenvalues and eigenvectors with respect to parameter θ will be calculated, where

$$q_1 = \tfrac{3}{2}\theta - \tfrac{1}{2} \tag{103}$$

$$q_2 = \theta \tag{104}$$

At $q_1 = q_2 = 1 \Leftrightarrow \theta = 1$ the operators of the partial differentiation are related via

$$\frac{\partial}{\partial \theta} = \frac{3}{2}\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} \tag{105}$$

$$\frac{\partial^2}{\partial \theta^2} = \frac{9}{4}\frac{\partial^2}{\partial q_1^2} + 3\frac{\partial^2}{\partial q_1 \partial q_2} + \frac{\partial^2}{\partial q_2^2} \tag{106}$$

Using the first operator equation together with Eqs. (95) and (96) leads to

$$\Lambda_{,\theta} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{107}$$

From Eq. (102) the second operator equation yields

$$\Lambda_{,\theta,\theta} = \frac{4}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{108}$$

To calculate the derivatives of the eigenvectors with respect to θ at $\theta = 1$, the operator in Eq. (105) leads to

$$X_{,\theta} = \frac{3}{2}XZ_1 = \begin{bmatrix} 0 & -2b/3 & 2c/3 \\ 0 & -b/3 & -2c/3 \\ 0 & -2b/3 & 2c/3 \end{bmatrix} \tag{109}$$

The results given by Friswell[5] match the results in Eqs. (107–109).

V. Conclusions

The existence of the derivatives of eigenvalues is investigated. In the case of multiple eigenvalues, the partial derivatives do not exist for any parameterization. Two conditions are deduced, which enable a given parameterization to be tested to determine its permissability. For any continuous permissible parameterization, the partial derivatives of the eigenvalues and eigenvectors are continuous, too. The application of the theorems presented has been demonstrated by examples. In preparation is a method to calculate the partial derivatives of repeated eigenvalues independent of the existence of eigenvectors.

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References

[1]Chen, T.-U., "Design Sensitivity Analysis for Repeated Eigenvalues in Structural Design," *AIAA Journal*, Vol. 31, No. 12, 1993, pp. 2347–2350.

[2]Choi, K. K., and Haug, E. J., "Optimization of Structures with Repeated Eigenvalues," *Optimization of Distributed Parameter Structures*, edited by E. J. Haug and J. Cea, Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands, 1981, pp. 219–277.

[3]Choi, K. K., and Haug, E. J., "A Numerical Method for Optimizing Structures with Repeated Eigenvalues," *Optimization of Distributed Parameter Structures*, edited by E. J. Haug and J. Cea, Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands, 1981, pp. 534–551.

[4]Choi, K. K., and Haug, E. J., "Repeated Eigenvalues in Mechanical Optimization Problems," *Problems of Elastic Stability and Vibrations*, edited by V. Komkov, Vol. 4, Contemporary Mathematics, American Mathematical Society, 1981, pp. 61–86.

[5]Friswell, M. I., "The Derivatives of Repeated Eigenvalues and Their Associated Eigenvectors," *Journal of Vibration and Acoustics*, Vol. 118, July 1996, pp. 390–397.

[6]Haug, E. J., and Choi, K. K., "Systematic Occurrence of Repeated Eigenvalues in Structural Optimization," *Journal of Optimization Theory and Applications*, Vol. 38, No. 2, 1982, pp. 251–274.

[7]Haug, E. J., Choi, K. K., and Komkov, V., *Design Analysis of Structural Systems*, Academic, New York, 1986.

[8]Haug, E. J., and Rousselet, B., "Design Sensitivity Analysis in Structural Mechanics, II, Eigenvalue Variations," *Journal of Structural Mechanics*, Vol. 2, No. 8, 1980, pp. 161–186.

[9]Lallement, G., and Kozanek, J., "Parametric Correction of Self-Adjoint Finite Element Models in the Presence of Multiple Eigenvalues," *Inverse Problems in Engineering*, Vol. 1, 1995, pp. 107–131.

[10]Masur, E. F., and Mróz, Z., "Non-Stationary Optimality Conditions in Structural Design," *International Journal of Solid Structures*, Vol. 15, No. 6, 1979, pp. 503–512.

[11]Masur, E. F., and Mróz, Z., "Singular Solutions in Structural Optimization Problems," *Variational Methods in the Mechanics of Solids*, edited by S. Nemat-Nesser, Pergamon, New York, 1980, pp. 337–343.

[12]Mottershead, J. E., and Friswell, M. I., "Model Updating in Structural Dynamics: A Survey," *Journal of Sound and Vibration*, Vol. 167, No. 2, 1993, pp. 347–375.

[13]Seyranian, A. P., Lund, E., and Olhoff, N., "Multiple Eigenvalues in Structural Optimization Problems," *Structural Optimization*, Vol. 8, Dec. 1994, pp. 207–227.

[14]Curtis, M. L., *Matrix Groups*, Springer–Verlag, New York, 1979.

[15]Humphreys, J. E., *Introduction to Lie Algebras and Representation Theory*, Springer–Verlag, New York, 1972.

[16]Jacobson, N., *Basic Algebra*, H. W. Freeman, San Francisco, CA, 1974.

[17]Rao, R. C., and Mitra, S. K., *Generalized Inverse of Matrices and its Applications*, Wiley, New York, 1974.

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